Iterated Limits in General Analysis.

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Introduction.

In a former note* we have briefly indicated a method for the investigation of iterated limits of functions on an abstract range. It is the purpose of the present paper to give a more comprehensive account of the method there proposed. The paper has its origin in the thought that in most of the definitions of limit that are employed in current mathematics a notion analogous to that of "neighborhood" or "vicinity" of an element is fundamental. In the domain of general analysis† various ways of determining a neighborhood of an element have been employed, notably the notion of *voisinage* used by M. Fréchet,‡ and the relations K_1 and K_2 used by E. H. Moore, either as undefined or as defined in terms of a "development" of the class of elements constituting the fundamental domain.§

A definite class of elements being assumed, the notion of "neighborhood" of an element is essentially that of a subclass having a special relation to the element. In taking this relation as undefined and at the basis of our system of postulates we occupy a position intermediate, as regards generality, between the extreme position of those who take the notion of "limit" itself as undefined,‖ and that of those who define "limit" by means of other relations which give rise to notions analogous to that of "neighborhood." The character and form of the postulates adopted are determined largely by two fundamental require-

* *Bull. Am. Math. Soc.*, Vol. XVII (July, 1911), p. 538.

† The term "general analysis" is here used in a technical sense to indicate mathematical analysis pertaining to a class of elements whose character is not specified.

‡ "Sur quelques points du calcul fonctionnel," *Rendiconti del Circolo Matematico di Palermo*, Vol. XXII.

§ E. H. Moore, "Introduction to a Form of General Analysis," pp. 125 and 138.

‖ For example, Fréchet in the first chapter of the paper referred to above, and F. Riesz in his paper before the International Congress of Mathematicians at Rome, 1908 ("Stetigkeitsbegriff und abstracte Mengenlehre," *Atti*, Vol. II, pp. 18–24).

ments;* first, to provide for an adequate treatment of ideal limiting elements, and second, to insure the persistence of the specified conditions under composition of ranges.

In Chapter I, we consider a class $\mathfrak{P}$ of elements and an undefined relation R between subclasses of $\mathfrak{P}$, the system $(\mathfrak{P}; R)$ being subjected to a set of postulates that permit the definition of ideal elements in such fashion that the system, when once extended by the adjunction of ideal elements, is closed to further extension in this manner. It is shown also that from two or more systems a composite system may be derived, and that the composite system satisfies the postulates if and only if the postulates are satisfied by every component system.

A somewhat less restrictive body of postulates, considered in Chapter II, pertain to a system $(\mathfrak{P}; \mathfrak{U}; T)$, $\mathfrak{P}$ being a class of elements, $\mathfrak{U}$ a class of ideal elements, and T a relation between subclasses of $\mathfrak{P}$ and individual elements of $\mathfrak{P}$ or $\mathfrak{U}$. A subclass $\mathfrak{R}$ of $\mathfrak{P}$ having the relation T to an element p of $\mathfrak{P}$ or to an ideal element u of $\mathfrak{U}$ may be thought of as a generalized neighborhood of p or u. The postulates of Chapter I, with the definition of ideal elements for the system $(\mathfrak{P}; R)$, lead to a system satisfying the postulates of Chapter II. We obtain for our system a generalization of a portion of the theory of point-sets by establishing relations between our postulates and the more general conditions involved in the notion of "limit" as used by Fréchet, and those involved in the "Verdichtungstelle" of F. Riesz.

In the third chapter a system $(\overline{\mathfrak{P}}; \mathfrak{U}; T)$ is supposed to satisfy the postulates of Chapter II, and functions μ defined on the range $\mathfrak{P}$, a subclass of $\overline{\mathfrak{P}}$, are studied relative to limits and continuity. The treatment is not intended to be exhaustive, the theorems developed being such as are suggested by familiar theorems on multiple sequences and functions of real variables. Interesting features of the general theory associate themselves with the presence of ideal elements in the system, and with the study of a property of functions which has much the same force as uniform continuity, but which we have called *extensible* continuity.

The fourth chapter is given to applications of the general theory through direct specialization of the system and particular determination of other arbitrary features. Special systems $(\mathfrak{P}; \mathfrak{U}; T)$ are specified, by consideration of

* E. R. Hedrick (*Transactions*, Vol. XII (1911), p. 289) obtains by his "inclosable" property of the fundamental domain essentially a generalization of the notion "neighborhood," but his assumptions are made from a different point of view and, involving a certain uniformity, are more restrictive than the postulates of the present paper.

which the theorems of Chapter III pertain to: The theory of multiple sequences; functions of real variables; functions on a range for which there is defined a relation of the type of either of the relations K_1 and K_2 used by Professor Moore; functions on a range subject to the *voisinage* used by Fréchet; and functions on a range whose elements are real-valued functions on an arbitrary range. In some cases the system $(\mathfrak{P};\, \mathfrak{U};\, T)$ is reached by the mediation of a system $(\mathfrak{P};\, R)$, and in some cases directly. In the applications, under certain restrictions on the class $\mathfrak{P}$, the property extensible continuity is found to be equivalent to uniform continuity in each case where the latter is defined.

We find it advantageous to draw largely upon the notation and terminology used by Professor Moore in his work on General Analysis. Convenience and economy of notation are conserved by the adoption of letters for elements, classes, etc., whose connotation renders frequent explanatory remarks unnecessary. Classes of elements are denoted by $\mathfrak{P}, \mathfrak{Q}, \mathfrak{R}$, etc., while their elements are denoted by p, q, r, etc., respectively. Classes of classes are, in general, denoted by u, v, w, etc.; properties and relations by P, Q, R, etc., or simply by the numerals attached to their definitions. Superscripts denote, in general, defining properties or conditions, the character of the superscript as well as of the base symbol serving to determine the nature of the limitation. Thus, $\mathfrak{R}^{\mathfrak{P}}$ states that $\mathfrak{R}$ is a subclass of $\mathfrak{P}$, $p^{\mathfrak{R}}$ that p is an element of $\mathfrak{R}$, $\mathfrak{P}^{P}$ that $\mathfrak{P}$ has the property P, etc. The symbol $\supset$ is a sign of implication, to be used in the statement of a proposition. That which precedes the sign of implication is hypothesis or given data, and that which follows is conclusion or a true statement concerning the given data. Thus, if A and B are propositions, $A \,.\!\supset\!.\, B$ is read "A implies B" or "if A then B," and if x represents a number in a certain interval and F a definite function on the interval, the proposition "for every two numbers x_1 and x_2 of the interval $F(x_1) - F(x_2) < k$" may be written, $x_1 \,.\, x_2 \,.\!\supset\!.\, F(x_1) - F(x_2) < k$. The reversed symbol $\subset$ denotes "is implied by" and $\backsim$ is the symbol of logical equivalence, "implies and is implied by." In a complex statement the symbols $\supset$, $\subset$ and $\backsim$ carry punctuation marks, ., :, .:, etc., the primary implication of the proposition being indicated by the greater number of dots. The mark $\exists$ is read "there exists," and the mark $\ni$ may be read "such that" or "where" as the sense of the proposition demands.

The independent use of the symbolical statement of propositions is confined largely to the proofs of theorems, where it is most useful in conserving precision and brevity, and where the technical symbols may be least objectionable to the general reader.

CHAPTER I.

THE SYSTEM $(\mathfrak{P}; R)$: EXTENSION AND COMPOSITION OF SYSTEMS.

§ 1. *Introductory: The System* $(\mathfrak{P}; R)$.

In this chapter we consider a system $(\mathfrak{P}; R)$ consisting of a class $\mathfrak{P}$ of elements p and a relation R on ordered pairs of subclasses of $\mathfrak{P}$. While the relation R is of the definite type indicated, it is not further specifically defined. We specify a system $(\mathfrak{P}; R)$ by specifying the class $\mathfrak{P}$ and the relation R, *i. e.*, a criterion which determines for every two subclasses $\mathfrak{R}_1$ and $\mathfrak{R}_2$ of $\mathfrak{P}$ whether or not $\mathfrak{R}_1$ has the relation R to $\mathfrak{R}_2$.

For example, take for $\mathfrak{P}$ the class of all points of an ordinary Euclidean plane. Consider a circle as the class of all points within and on its circumference, then we may specify a relation R in terms of geometry as follows: Every circle whose radius is different from zero has the relation R to the point at its center considered as singular subclass, and every two concentric circles whose radii are different from zero have the relation R to each other. In no other case does the relation R hold.

In this example we have a definite system $(\mathfrak{P}; R)$. The pertinence of the relation R as specified to the study of limits of functions defined for a set of points in the plane is obvious. A study of the current theory of real-valued functions, in particular in connection with questions of continuity and iterated limits, leads to a determination of bodies of postulates on systems $(\mathfrak{P}; R)$ which serve to validate a theory of continuous functions and multiple and iterated limits associated with such systems $(\mathfrak{P}; R)$ in general.

Subclasses of $\mathfrak{P}$ are, in general, denoted by $\mathfrak{R}$, and the notation $\mathfrak{R}_1 R \mathfrak{R}_2$ indicates that $\mathfrak{R}_1$ has the relation R to $\mathfrak{R}_2$, while $\mathfrak{R}_1 {}^- R \mathfrak{R}_2$ indicates that $\mathfrak{R}_1$ does not have the relation R to $\mathfrak{R}_2$. In case it is desired to imply that a subclass consists of a single element, we may for simplicity, and for our purposes without confusion, use the notation for single elements. Thus $\mathfrak{R} R p$ indicates that the class $\mathfrak{R}$ has the relation R to the singular subclass whose element is p. The letter v denotes a class of subclasses $\mathfrak{R}$ of $\mathfrak{P}$, and, for a given element p, v_p is the class of all subclasses $\mathfrak{R}$ having the relation R to p, *i. e.*,

$$v_p = [\text{all } \mathfrak{R} \ni \mathfrak{R} R p].$$

Thus, in the example above, v_p is the family of concentric circles whose common center is at the point p, excluding the point circle of the family.

§ 2. *The Postulates and Certain Fundamental Definitions.*

Preliminary to the statement of postulates for a system $(\mathfrak{P}; R)$, we note that a class v of subclasses $\mathfrak{R}$ of $\mathfrak{P}$ may have one or more of the following properties:

1. Every member $\mathfrak{R}$ of the class v contains at least one element p.

2. The relation R holds between every two classes $\mathfrak{R}_1$ and $\mathfrak{R}_2$ that are members of v.

3. There exists a sequence $\{\mathfrak{R}_n\}$ of members of the class v such that for every $\mathfrak{R}$ of v there is a number $n_{\mathfrak{R}}$ such that for $n > n_{\mathfrak{R}}$ the class $\mathfrak{R}_n$ is contained in $\mathfrak{R}$.

4. For every $\mathfrak{R}$ of v there exists an $\mathfrak{R}_1$ of v such that for every p in $\mathfrak{R}_1$ there is a subclass $\mathfrak{R}_2$ of $\mathfrak{R}$ having the relation R to the singular class p.

5. If v_1 is a class containing v and having properties 1, 2, 3 and 4, then $v = v_1$.

6. If v_1 is a class having properties 1, 2, 3, 4 and 5, and not containing v, then there exists a member $\mathfrak{R}_1$ of v_1 and a member $\mathfrak{R}_2$ of v such that $\mathfrak{R}_1$ and $\mathfrak{R}_2$ have no common elements.

7. For every element p of $\mathfrak{P}$ there is an $\mathfrak{R}$ of v which does not contain p.

These definitions may be more concisely stated in symbols as follows:

1. $\mathfrak{R}^v \,.\supset.\, \exists\, p^{\mathfrak{R}}.$
2. $\mathfrak{R}_1^v \,.\, \mathfrak{R}_2^v \,.\supset.\, \mathfrak{R}_1 R \mathfrak{R}_2 .$
3. $\exists\, \{\mathfrak{R}_n\} \ni [(n \,.\supset.\, \mathfrak{R}_n^v) \,.\, (\mathfrak{R}^v :\supset: \exists\, n_{\mathfrak{R}} \ni n > n_{\mathfrak{R}} \,.\supset.\, \mathfrak{R}_n^{\mathfrak{R}})].$
4. $\mathfrak{R}^v :\supset: \exists\, \mathfrak{R}_1^v \ni p^{\mathfrak{R}_1} \,.\supset.\, \exists\, \mathfrak{R}_2^{\mathfrak{R}} \ni \mathfrak{R}_2 R\, p.$
5. $v^{v_1} \,.\, v^{1.2.3.4} \,.\supset.\, v = v_1 .$
6.* $v_1^{1.2.3.4.5} \,.\, v^{-v_1} \,.\supset.\, \exists\, (\mathfrak{R}_1^{v_1} \,.\, \mathfrak{R}_2^v) \ni {}^{-}\exists\, p \ni p^{\mathfrak{R}_1} \,.\, p^{\mathfrak{R}_2} .$
7. $p \,.\supset.\, \exists\, \mathfrak{R}^v \ni p^{-\mathfrak{R}} .$

These properties, 1–7, may be called propositional properties.† It is not here asserted that any of the defining propositions are true with respect to any class v, but it is clear that the question whether or not a given one of these propositions is true with respect to a given class v is a question of the presence or absence of a definite property for the class.

The desired postulates might now be stated in the following form:

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* The minus sign here signifies negation. Thus $^{-}\exists$ is read "there does not exist," and $p^{-\mathfrak{R}}$ indicates that p is not an element of the subclass $\mathfrak{R}$. v^{-v_1} indicates that v is not a subclass of v_1.

† See E. H. Moore, *loc. cit.*, p. 20.

(A) *For every element p the class v_p has properties* 1–6, *i. e.*,

$$p \,.\supset.\, v_p^{1.2.3.4.5.6}.$$

(B) *For every element p it is true that every $\mathfrak{R}$ of v_p contains p, while if p_1 is distinct from p there is an $\mathfrak{R}$ of v_p not containing p_1.* In symbols:

$$p :\supset: (\mathfrak{R}\,R\,p \,.\supset.\, p^{\mathfrak{R}}) \,.\, (p_1 \neq p \,.\supset.\, \exists\, \mathfrak{R} \ni \mathfrak{R}\,R\,p \,.\, p_1^{-\mathfrak{R}}).$$

But for convenience of reference, as well as to provide for discussion of the independence of the conditions on the system, we separate these assumptions into simpler components, which we state explicitly in the following seven postulates:

I. $\mathfrak{R}\,R\,p \,.\supset.\, p^{\mathfrak{R}}$.

II. $\mathfrak{R}_1\,R\,p \,.\, \mathfrak{R}_2\,R\,p \,.\supset.\, \mathfrak{R}_1\,R\,\mathfrak{R}_2$.

III. $p \,.:\supset:.\, \exists\, \{\mathfrak{R}_n\} \ni [(n \,.\supset.\, \mathfrak{R}_n R\,p) \,.\, (\mathfrak{R}\,R\,p :\supset: \exists\, n_{\mathfrak{R}} \ni n > n_{\mathfrak{R}} \,.\supset.\, \mathfrak{R}_n^{\mathfrak{R}})]$.

IV. $\mathfrak{R}\,R\,p :\supset: \exists\, \mathfrak{R}_1 \ni [\mathfrak{R}_1\,R\,p \,.\, (p_1^{\mathfrak{R}_1} \,.\supset.\, \exists\, \mathfrak{R}_2^{\mathfrak{R}} \ni \mathfrak{R}_2\,R\,p_1)]$.

V. $v^{1.2.3.4} \,.\, (\mathfrak{R}\,R\,p \,.\supset.\, \mathfrak{R}^{v}) :\supset: v = v_p$.

VI. $v^{1.2.3.4.5} \,.\, v_p^{-v} \,.\supset.\, \exists\, (\mathfrak{R}_1^{v} \,.\, \mathfrak{R}_2^{v_p}) \ni -\exists\, p_1 \ni p_1^{\mathfrak{R}_1} \,.\, p_1^{\mathfrak{R}_2}$.

VII. $p_1 \neq p \,.\supset.\, \exists\, \mathfrak{R} \,\varepsilon\, \mathfrak{R}\,R\,p \,.\, p_1^{-\mathfrak{R}}$.

Postulates I and VII are together equivalent to the statement (B). A corollary of postulate I is that for every p the class v_p has property 1, while postulates II–VI state that for every p the class v_p has the respective properties 2–6.

The following examples are pertinent to the question of consistency and independence of the postulates. Example 0 is an instance of a system satisfying the seven postulates, and the remaining examples each violate one postulate and satisfy all the others, the examples being numbered in the order of the postulates violated.

Ex. 0. The class $\mathfrak{P}$ is the class of all complex numbers. The notation $\mathfrak{R}_{dp}$, where d is a positive real number and p is an element of $\mathfrak{P}$, stands for the subclass of $\mathfrak{P}$ consisting of all elements p_1 of $\mathfrak{P}$ such that $|p_1 - p| \leqq d$, that is,

$$\mathfrak{R}_{dp} \equiv [\text{all } p_1 \ni |p_1 - p| \leqq d].$$

The relation R is specified as follows: For every p and every d the relation $\mathfrak{R}_{dp}Rp$ holds, and for every p and every d_1 and d_2 the relation $\mathfrak{R}_{d_1 p}\,R\,\mathfrak{R}_{d_2 p}$ holds. The relation holds in no other case.

Ex. 1. The system $(\mathfrak{P}; R)$ is specified as in example 0, except that $\mathfrak{R}_{dp}$ does not contain the element p, hence

$$R_{dp} \equiv [\text{all } p_1 \neq p \ni |\, p_1 - p\,| \leqq d].$$

Ex. 2. The class $\mathfrak{P}$ is the class of all complex numbers, and the notation $\mathfrak{R}_{dp}$ has the same significance as in example 0. For every p and for every d the relation $\mathfrak{R}_{dp} R p$ holds, but in no other case does the relation R hold.

In this example postulates V and VI are satisfied vacuously, *i. e.*, their hypotheses are incapable of fulfilment, there being no class v which has properties 2 and 3.

Ex. 3. The class $\mathfrak{P}$ is the class of all points of a given Euclidean plane. The designation "line" is used to indicate a subclass $\mathfrak{R}$ constituting the class of all points of a line. Every "line" has the relation R to every one of its points, and every two intersecting "lines" have the relation R to each other. In no other case does the relation R hold.

Here "intersecting" is interpreted as "having a point in common," so that a "line" has the relation R to itself. Postulates V and VI are again satisfied vacuously.

Ex. 4. The system $(\mathfrak{P}; R)$ is as specified in example 0, except that in the particular case $p = 0$ the classes R_{dp} consist only of real elements p_1, *i. e.*,

$$\mathfrak{R}_{d0} \equiv [\text{all real } p_1 \ni |p_1| \leqq d].$$

Ex. 5. Again, the class $\mathfrak{P}$ is the class of all complex numbers, and the notation $\mathfrak{R}_{dp}$ has the same significance as in example 0. The relation R is as specified in example 0 except that for the particular element $p_0 = 0$ the relation $\mathfrak{R}_{dp_0} R p_0$ holds only in case d is less than or equal to unity.

Ex. 6. The class $\mathfrak{P}$ consists of two elements, p_1 and p_2.* The cases in which the relation R holds are listed as follows:

$$p_1 R p_1, \quad p_2 R p_2, \quad \mathfrak{P} R \mathfrak{P}.$$

Ex. 7. Again, $\mathfrak{P}$ is a class consisting of two elements, p_1 and p_2. Following is the list of cases in which the relation R holds:

$$\mathfrak{P} R p_1, \quad \mathfrak{P} R p_2, \quad \mathfrak{P} R \mathfrak{P}.$$

In this instance postulate VI is satisfied vacuously, since any class v that has properties 1–5 possesses the single member $\mathfrak{P}$, and is therefore coincident with both v_{p_1} and v_{p_2}.

* It should be remembered that elements do not enter in the relation R. The notation for elements is substituted for class notation as a matter of convenience. The class v_{p_1} consists of one member, the class $\mathfrak{R}$ having the single element p_1, and has no member in common with that class v whose only member is $\mathfrak{P}$.

§ 3. *Extension of the System by the Adjunction of Ideal Elements*

Making use of properties 1–7 defined in § 2, properties that may be possessed by a class v of subclasses $\mathfrak{R}$ of $\mathfrak{P}$, we proceed to the definition of ideal elements for the system $(\mathfrak{P}; R)$.

Def. 1. An *ideal element* of the system $(\mathfrak{P}; R)$ is a class v of subclasses $\mathfrak{R}$ of $\mathfrak{P}$ having properties 1–7.

The letter u invariably stands for an ideal element.

THEOREM I. *If v is a class having properties* 1–6, *then v is an ideal element u, or there is an element p such that $v = v_p$.*

Proof: If v has property 7, it is a u by definition; if it has not property 7, then there is an element p common to all classes $\mathfrak{R}$ of v. Since v_p has properties 1–6, we clearly have $v = v_p$.

Let $\mathfrak{U}$ denote the class of all ideal elements of the system $(\mathfrak{P}; R)$, and let $\mathfrak{Q}$ be a class consisting of the elements of $\mathfrak{P}$, together with all ideal elements, *i. e.*, $\mathfrak{Q} \equiv \mathfrak{P} + \mathfrak{U}$. We denote elements of $\mathfrak{Q}$, in general, by q, subclasses of $\mathfrak{Q}$ by $\mathfrak{S}$, and classes of subclasses by w. A certain technical form of correspondence between classes is of frequent occurrence, and it is therefore convenient to adopt a special symbol, $\|$, to be read *corresponds to*, which we define as follows:

Def. 2. $\mathfrak{S} \| \mathfrak{R}$ indicates that $\mathfrak{S}$ consists of the elements of $\mathfrak{R}$ together with every ideal element u such that there is a subclass $\mathfrak{R}_1$ of $\mathfrak{R}$ which belongs to the class* u. In symbols:

$$\mathfrak{S} \| \mathfrak{R} : \equiv : \mathfrak{S} = \mathfrak{R} + [\text{all } u \ni \exists\, \mathfrak{R}_1^{\mathfrak{R}} \ni \mathfrak{R}_1^u].$$

Def. 3. $w \| v$ indicates that w consists of all classes $\mathfrak{S}$ for which there exist classes $\mathfrak{R}$ in v such that $\mathfrak{S} \| \mathfrak{R}$. In symbols:

$$w \| v : \equiv : w = [\text{all } \mathfrak{S} \ni \exists\, \mathfrak{R}^v \ni \mathfrak{S} \| \mathfrak{R}].$$

It is obvious that for every $\mathfrak{R}$ there is a unique $\mathfrak{S}$ such that $\mathfrak{S} \| \mathfrak{R}$, and that for two distinct classes $\mathfrak{R}_1$ and $\mathfrak{R}_2$ the corresponding classes $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are distinct. It follows that for every v there is a unique w, and that for two distinct classes v_1 and v_2 the corresponding w_1 and w_2 are distinct.

Let S be a relation of the same type as R defined as follows:

Def. 4. The relation $\mathfrak{S}_1 S \mathfrak{S}_2$ holds if and only if one of the following conditions is fulfilled:

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* No confusion need arise from the fact that the letter u denotes at the same time an element of $\mathfrak{Q}$ and a class of subclasses of $\mathfrak{Q}$, as well as of $\mathfrak{P}$. It was this double rôle that led to the adoption of small letters as notation for classes of subclasses in general.

(a) $\exists\ (\mathfrak{R}_1 \,.\, \mathfrak{R}_2) \ni \mathfrak{R}_1 R\, \mathfrak{R}_2 \,.\, \mathfrak{S}_1 \,\|\, \mathfrak{R}_1 \,.\, \mathfrak{S}_2 \,\|\, \mathfrak{R}_2 \,.$

(b) $\exists\ (\mathfrak{R} \,.\, u) \ni \mathfrak{R}^u \,.\, \mathfrak{S}_1 \,\|\, \mathfrak{R} \,.\, (q^{\mathfrak{S}_2} \,.\supset.\, q = u).$

In condition (b) the ideal element u considered as a class of subclasses contains $\mathfrak{R}$, and considered as an element of $\mathfrak{Q}$ constitutes the singular subclass $\mathfrak{S}_2$.

We now have a definite system $(\mathfrak{Q}; S)$, which we shall call the *extended system* derived from $(\mathfrak{P}; R)$. We investigate the character of this extended system with respect to the seven postulates. The properties 1–7 defined for a class v are defined also for a class w, if in the notation we replace v by w, p by q, $\mathfrak{R}$ by $\mathfrak{S}$ and R by S, and with similar changes of notation we have the seven postulates stated for the system $(\mathfrak{Q}; S)$.

THEOREM II. *The seven postulates are satisfied by the extended system* $(\mathfrak{Q}; S)$.

In proving this theorem it it convenient to establish first the following lemma:

LEMMA. *The necessary and sufficient condition that a class w shall have properties* 1–4 *or* 1–5 *or* 1–6 *is that there shall exist a class v having the corresponding properties such that $w \,\|\, v$.*

In considering the necessity of the condition we have available in each case the fact that w has property 2. This is sufficient to secure the existence of a v such that $w \,\|\, v$. It is sufficient, then, to assume a definite w corresponding to a definite v and prove the following propositions:

(a)	$w^{1.2.3.4} \,.\supset.\, v^{1.2.3.4}$,	(b)	$v^{1.2.3.4} \,.\supset.\, w^{1.2.3.4}$,
(c)	$w^{1.2.3.4.5} \,.\supset.\, v^{5}$,	(d)	$v^{1.2.3.4.5} \,.\supset.\, w^{5}$,
(e)	$w^{1.2.3.4.5.6} \,.\supset.\, v^{6}$,	(f)	$v^{1.2.3.4.5.6} \,.\supset.\, w^{6}$.

(a): Since w has property 1, every $\mathfrak{S}$ of w contains a q, that is, either a p or a u. Every $\mathfrak{R}$ of v has a corresponding $\mathfrak{S}$ in w, hence it contains either this element p or an $\mathfrak{R}_1$ of this class u. Therefore v has property 1. That v has property 2 is evident from definition 4, and from property 3 of w there is a sequence $\{\mathfrak{S}_n\}$ such that the sequence $\{\mathfrak{R}_n\}$, where $\mathfrak{S}_n \,\|\, \mathfrak{R}_n$, is effective in establishing property 3 for v. Since w has property 4, we have

$$(1)\qquad \mathfrak{S}^w :\supset: \exists\ \mathfrak{S}_1^w \ni q^{\mathfrak{S}_1} \,.\supset.\, \exists\ \mathfrak{S}_2^{\mathfrak{S}} \ni \mathfrak{S}_2 S q,$$

and we wish to prove

$$(2)\qquad \mathfrak{R}^v :\supset: \exists\ \mathfrak{R}_1^v \ni p^{\mathfrak{R}_1} \,.\supset.\, \exists\ \mathfrak{R}_2^{\mathfrak{R}} \ni \mathfrak{R}_2 R p.$$

Given an $\mathfrak{R}$ of v, take $\mathfrak{S}$ such that $\mathfrak{S} \parallel \mathfrak{R}$, then $\mathfrak{S}^w$ and (1) applies. Take $\mathfrak{R}_1$ so that $\mathfrak{S}_1 \parallel \mathfrak{R}_1$, then $\mathfrak{R}_1^v$ and every p of $\mathfrak{R}_1$ is in $\mathfrak{S}_1$. For every p of $\mathfrak{R}_1$, then, there is a subclass $\mathfrak{S}_2$ of $\mathfrak{S}$ such that $\mathfrak{S}_2 S p$, and there is an $\mathfrak{R}_2$ such that $\mathfrak{S}_2 \parallel \mathfrak{R}_2$, then clearly $\mathfrak{R}_2^{\mathfrak{R}}$ and $\mathfrak{R}_2 R p$. Thus v has property 4.

(b): The proof is obvious for properties 1, 2 and 3. As to property 4, we have condition (2) above and wish to prove (1). Given an $\mathfrak{S}$ of w, there is a corresponding $\mathfrak{R}$ in v, so that (2) applies to provide an $\mathfrak{R}_1$ fulfilling the conclusion of (2). Take $\mathfrak{S}_1$ to correspond to $\mathfrak{R}_1$, then $\mathfrak{S}_1^w$ and every p in $\mathfrak{S}_1$ is in $\mathfrak{R}_1$ and an $\mathfrak{S}_2$ corresponding to an $\mathfrak{R}_2$ furnished by (2) meets the requirements of (1). Further, every u in $\mathfrak{S}_1$ possesses a member $\mathfrak{R}_2$ which is a subclass of $\mathfrak{R}_1$, and since $\mathfrak{R}_1$ is necessarily a subclass of $\mathfrak{R}$, we see that the class $\mathfrak{S}_2$ corresponding to $\mathfrak{R}_2$ is a subclass of $\mathfrak{S}$; and clearly $\mathfrak{S}_2 S u$, therefore w has property 4.

Propositions (c) and (d) are easily verified by the use of (a) and (b).

(e): From property 6 of w we have

$$(3) \qquad w_1^{1.2.3.4.5} \cdot w^{-w_1} \cdot \supset \cdot \exists\, (\mathfrak{S}_1^{w_1} \cdot \mathfrak{S}_2^{w}) \ni -\exists\, q \ni q^{\mathfrak{S}_1} \cdot q^{\mathfrak{S}_2},$$

and we wish to prove

$$(4) \qquad v_1^{1.2.3.4.5} \cdot v^{-v_1} \cdot \supset \cdot \exists\, (\mathfrak{R}_1^{v_1} \cdot \mathfrak{R}_2^{v}) \ni -\exists\, p \ni p^{\mathfrak{R}_1} \cdot p^{\mathfrak{R}_2}.$$

If v_1 has properties 1–5 and does not contain v, then there is a w_1 such that $w_1 \parallel v_1$ which does not contain w and which, by (b) and (d), has properties 1–5. Proposition (3) is now applicable, and the $\mathfrak{S}_1$ and $\mathfrak{S}_2$ thus available have corresponding classes $\mathfrak{R}_1$ and $\mathfrak{R}_2$ which obviously fulfil the conclusion of (4).

The proof of (f) is similar to that of (e).

The proof of the theorem is now easily completed. In analogy with previous notation, we denote by w_q the class of all classes $\mathfrak{S}$ such that $\mathfrak{S} S q$, and we observe that for every p we have $w_p \parallel v_p$, while for every u we have $w_u \parallel u$. Since every v_p and every u have properties 1–6, it follows from the lemma that the class w_q has properties 1–6. Postulate I being obviously fulfilled, it remains to consider postulate VII. We wish to show that

$$q_1 \neq q_2 \cdot \supset \cdot \exists\, \mathfrak{S} \ni (\mathfrak{S} S q_1 \cdot q_2^{-\mathfrak{S}}).$$

If both q_1 and q_2 are elements of $\mathfrak{P}$, postulate VII on the system $(\mathfrak{P}; R)$ assures us of a class $\mathfrak{R}$ such that the corresponding class $\mathfrak{S}$ is effective. If either q_1 or q_2 is a u, then, since by property 7 of u and postulate I the class u is not contained in any class v_p, the corresponding class w_u is not contained in

any class w_p, and therefore property 6 of w_u is effective, the desired conclusion being an immediate consequence.

It is desirable now to show that the extended system $(\mathfrak{Q}; S)$ is closed to this process of extension; that is, if we repeat the process of extension the second extended system coincides with the first. We may state the required theorem in the form:

THEOREM III. *No ideal elements arise in the extended system* $(\mathfrak{Q}; S)$.

Proof: Suppose a class w to have properties 1–6, then there is a class v having properties 1–6 such that $w \parallel v$. By theorem I v is a u or a v_p, and in either case w is a w_q and therefore does not have property 7.

Following is an instance of a system which illustrates effectively the operation of the foregoing definition of extension:

Example. The class $\mathfrak{P}$ is the class of all rational numbers. If p_1 and p_2 are two distinct rational numbers, then, if p_1 is less than p_2, the class $\mathfrak{R}_{p_1 p_2}$ is the class of all rational numbers on the interval $p_1 p_2$. That is,

$$\mathfrak{R}_{p_1 p_2} \equiv [\text{all } p \ni p_1 \leqq p \leqq p_2].$$

The relation $\mathfrak{R}_{p_1 p_2} R \, \mathfrak{R}_{p_3 p_4}$ holds if and only if the intervals $p_1 p_2$ and $p_3 p_4$ have a common sub-interval, *i. e.*, if $p_1 < p_4$ and $p_3 < p_2$; and the relation $\mathfrak{R}_{p_1 p_2} R \, p$ holds if and only if $p_1 < p < p_2$. In no other case does the relation R hold.

It is not difficult to see that the system $(\mathfrak{P}; R)$ here specified satisfies the seven postulates. We proceed, therefore, to investigate the matter of ideal elements. Consider a class v having properties 1–6. By a little attention to the requirements of properties 3 and 4 we see that there exists a sequence $\{\mathfrak{R}_n\}$ of members of v such that for every n the class $\mathfrak{R}_n$ is of the form $\mathfrak{R}_{\underline{p}_n \bar{p}_n}$, where the sequence $\{\underline{p}_n\}$ is an increasing monotonic sequence of distinct elements and the sequence $\{\bar{p}_n\}$ is a decreasing monotonic sequence of distinct elements; and, further, such that every member $\mathfrak{R}$ of v contains a member $\mathfrak{R}_n$ of the sequence. In view of property 5, then, the two sequences $\{\underline{p}_n\}$ and $\{\bar{p}_n\}$ have a common limit. If this limit is a rational number p, then v coincides with v_p, while if the limit is an irrational number a, then v consists of all classes $\mathfrak{R}_{p_1 p_2}$ such that $p_1 < a < p_2$. In the latter case v has property 7 and is an ideal element of the system $(\mathfrak{P}; R)$. Since it is obvious that for every irrational number a the class v consisting of all classes $\mathfrak{R}_{p_1 p_2}$ such that $p_1 < a < p_2$ has properties 1–7, we see that the ideal elements u of the system $(\mathfrak{P}; R)$ are in reciprocal one-to-one correspondence with the irrational numbers in such fashion that, if u corresponds to a, then

$$u \equiv [\text{all } \mathfrak{R}_{p_1 p_2} \ni p_1 < a < p_2].$$

We may therefore consider our definition of ideal elements, in this instance, as a definition of irrational numbers.

The extended system $(\mathfrak{Q}; S)$ is seen to be as follows: $\mathfrak{Q}$ is the class of all real numbers; two intervals with rational limits that have a common interior element have the relation S to each other, and every interval with rational limits has the relation S to every one of its interior elements (considered as singular class), but in no other case does the relation S hold.

§ 4. *Composition of Systems.**

Two classes of elements, $\mathfrak{P}'$ and $\mathfrak{P}''$, determine a "product" or composite class, $\mathfrak{P} = \mathfrak{P}'\mathfrak{P}''$, consisting of all elements p of the form $p = (p', p'')$, where p' and p'' belong to the classes $\mathfrak{P}'$ and $\mathfrak{P}''$ respectively. It should be borne in mind that these bi-partite elements, $p = (p', p'')$, which we shall denote simply by $p'p''$, are not in any sense products of the elements p' and p'', but rather that p is a notation for the pair $p'p''$. From $\mathfrak{R}'$ and $\mathfrak{R}''$, subclasses of $\mathfrak{P}'$ and $\mathfrak{P}''$ respectively, we have $\mathfrak{R} = \mathfrak{R}'\mathfrak{R}''$, a subclass of $\mathfrak{P}$ consisting of all elements p of the form $p = p'p''$, where p' and p'' belong to $\mathfrak{R}'$ and $\mathfrak{R}''$ respectively. Similarly, if v' is a class of subclasses $\mathfrak{R}'$ of $\mathfrak{P}'$, and v'' is a class of subclasses $\mathfrak{R}''$ of $\mathfrak{P}''$, we have the composite class $v = v'v''$, the class of all $\mathfrak{R} = \mathfrak{R}'\mathfrak{R}''$, where $\mathfrak{R}'$ and $\mathfrak{R}''$ belong to the respective classes v' and v''.

If R' and R'' are relations of the type discussed in § 1, defined for $\mathfrak{P}'$ and $\mathfrak{P}''$ respectively, then from the two systems $(\mathfrak{P}'; R')$ and $(\mathfrak{P}''; R'')$ we derive what we shall call the composite of these two systems, the system $(\mathfrak{P}; R)$, where $\mathfrak{P} = \mathfrak{P}'\mathfrak{P}''$ and R is a relation of the same type as R' and R'' defined as follows:

Def. 4. For a system $(\mathfrak{P}; R)$, composite of $(\mathfrak{P}'; R')$ and $(\mathfrak{P}''; R'')$, the relation $\mathfrak{R}_1 R \mathfrak{R}_2$ holds if and only if there exist classes $\mathfrak{R}_1'$, $\mathfrak{R}_1''$, $\mathfrak{R}_2'$ and $\mathfrak{R}_2''$ such that $\mathfrak{R}_1 = \mathfrak{R}_1'\mathfrak{R}_1''$ and $\mathfrak{R}_2 = \mathfrak{R}_2'\mathfrak{R}_2''$, and such that the relations $\mathfrak{R}_1' R' \mathfrak{R}_2'$ and $\mathfrak{R}_1'' R'' \mathfrak{R}_2''$ hold.

It may be observed that the effectiveness of the foregoing definition is independent of the conditions imposed upon the component systems. It is convenient throughout the present section to regard the systems involved as unconditioned, except as conditions are specified in the hypotheses of the several theorems.

THEOREM IV. *The seven postulates are satisfied by the system* $(\mathfrak{P}; R)$, *composite of* $(\mathfrak{P}'; R')$ *and* $(\mathfrak{P}''; R'')$, *if and only if they are satisfied by both component systems.*

* Compare T. H. Hildebrandt, AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXIV (1912), p. 250.

We consider first the following lemma:

LEMMA I. *If* $v = v'v''$, *then the following propositions hold:*

(a) $v^{1.2.3.4} . \supset . v'^{1.2.3.4} . v''^{1.2.3.4}$, (b) $v'^{1.2.3.4} . v''^{1.2.3.4} . \supset . v^{1.2.3.4}$,

(c) $v^{1.2.3.4.5} . \supset . v'^{5} . v''^{5}$, (d) $v'^{1.2.4.4.5} . v''^{1.2.3.4.5} . \supset . v^{5}$,

(e) $v^{1.2.3.4.5.6} . \supset . v'^{6} . v''^{6}$, (f) $v'^{1.2.3.4.5.6} . v''^{1.2.3.4.5.6} . \supset . v^{6}$.

(a) and (b) are sufficiently obvious if only we remember the significance of the relation $v = v'v''$.

(c): Suppose a class v_1' to contain v' and to have properties 1–4. The composite class $v_1 = v_1'v''$ then contains v and, by (a) and (b), has properties 1–4; by property 5 of v then $v_1 = v$, and therefore $v_1' = v'$. In similar manner it may be shown that v'' has property 5.

(d): Suppose v_1 to contain v and to have properties 1–4, and consider the two classes v_1' and v_1'' defined as follows:

$$v_1' = [\text{all } \mathfrak{R}' \ni \exists\, (\mathfrak{R}'' . \mathfrak{R}^{v_1}) \ni \mathfrak{R} = \mathfrak{R}'\mathfrak{R}''],$$
$$v_1'' = [\text{all } \mathfrak{R}'' \ni \exists\, (\mathfrak{R}' . \mathfrak{R}^{v_1}) \ni \mathfrak{R} = \mathfrak{R}'\mathfrak{R}''].$$

It is not difficult to see that these classes have properties 1–4, and also that v_1' contains v' and v_1'' contains v''; then by property 5 of v' and v'', we have $v_1' = v'$ and $v_1'' = v''$, and consequently $v_1 = v$.

(e): Suppose v_1' has properties 1–5 and $v'^{-v_1'}$, then if $v_1 = v_1'v''$ we have $v_1^{1.2.3.4.5}$ and v^{-v_1}; hence by property 6 of v,

$$\exists\, (\mathfrak{R}_1^{v_1} . \mathfrak{R}_2^{v}) \ni -\exists\, p \ni p^{\mathfrak{R}_1} . p^{\mathfrak{R}_2}.$$

The classes $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are of the forms $\mathfrak{R}_1 = \mathfrak{R}_1'\mathfrak{R}_1''$ and $\mathfrak{R}_2 = \mathfrak{R}_2'\mathfrak{R}_2''$, where $\mathfrak{R}_1'$ and $\mathfrak{R}_2'$ are members of v_1' and v' respectively, and $\mathfrak{R}_1''$ and $\mathfrak{R}_2''$ are members of v''. By properties 1 and 3 of v'', $\mathfrak{R}_1''$ and $\mathfrak{R}_2''$ have common elements, therefore $\mathfrak{R}_1'$ and $\mathfrak{R}_2'$ can have no elements in common, and consequently v' has property 6. In like manner it may be shown that v'' has property 6.

(f): Suppose $v_1^{1.2.3.4.5}$ and v^{-v_1}; take v_1' and v_1'' defined as in the proof of (d) above, then as before they have properties 1–4, and by (b) the composite class $\bar{v}_1 = v_1'v_1''$ has these four properties. Since $\bar{v}_1$ clearly contains v_1, and v_1 has property 5, we have $\bar{v}_1 = v_1$; hence by (c) v_1' and v_1'' have property 5. Since v is not contained in v_1, either v' is not contained in v_1', or v'' is not contained in v_1''; suppose the former, then since v' has property 6, we have

$$\exists\, (\mathfrak{R}_1'^{v_1'} . \mathfrak{R}_2'^{v'}) \ni -\exists\, p' \ni p'^{\mathfrak{R}_1'} . p'^{\mathfrak{R}_2'}.$$

Take $\mathfrak{R}_1''$ and $\mathfrak{R}_2''$ so that $\mathfrak{R}_1''$ is a member of v_1'' and $\mathfrak{R}_2''$ is a member of v'', and

take $\mathfrak{R}_1 = \mathfrak{R}_1' \mathfrak{R}_1''$ and $\mathfrak{R}_2 = \mathfrak{R}_2' \mathfrak{R}_2''$; then $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are members of v_1 and v_2 respectively, and clearly they can have no common elements. Thus we show that v has property 6.

We record here for future reference certain results reached incidentally in the foregoing proofs:

LEMMA II. (a) *If v has properties* 1–4 *and*

$$v' = [\text{all } \mathfrak{R}' \ni \exists\, (\mathfrak{R}'' \,.\, \mathfrak{R}^v) \ni \mathfrak{R} = \mathfrak{R}' \mathfrak{R}''],$$

then v' has properties 1–4.

(b) *If v has properties* 1–5, *then there exist classes v' and v'' such that $v = v' v''$.*

Taking up now the proof of the theorem, we assume that the composite system $(\mathfrak{P}; R)$ satisfies the seven postulates and show* that as a consequence both component systems satisfy them. For a given p' there exist elements p and p'' such that $p = p' p''$ and $v_p = v'_{p'} v''_{p''}$, and since every $\mathfrak{R}$ of v_p contains p, clearly every $\mathfrak{R}'$ of $v'_{p'}$ contains p', so that postulate I is satisfied by $(\mathfrak{P}'; R')$. It follows also from lemma I that postulates II–VI are satisfied by this system. If $p_1' \neq p_2'$, then taking $p_1 = p_1' p''$ and $p_2 = p_2' p''$, where p'' is any element of $\mathfrak{P}''$, we have $p_1 \neq p_2$; hence there is an $\mathfrak{R}$ not containing p_2 such that $\mathfrak{R} R p_1$. But such an $\mathfrak{R}$ is of the form $\mathfrak{R} = \mathfrak{R}' \mathfrak{R}''$, where $\mathfrak{R}' R' p_1'$ and $\mathfrak{R}'' R'' p''$; and since p_1 is in $\mathfrak{R}$, we see that p_1' is in $\mathfrak{R}'$ and p'' is in $\mathfrak{R}''$. Clearly then, p_2' is not in $\mathfrak{R}'$, so that the system $(\mathfrak{P}'; R')$ satisfies postulate VII. In like manner the system $(\mathfrak{P}''; R'')$ is shown to satisfy the seven postulates.

The remainder of the theorem, *i. e.*, that if the postulates are satisfied by both component systems then they are satisfied by the composite system, is sufficiently evident without detailed discussion.

From the system $(\mathfrak{P}; R)$, composite of the two systems $(\mathfrak{P}'; R')$ and $(\mathfrak{P}''; R'')$, we may derive an extended system $(\mathfrak{Q}; S)$ by the process defined in § 3; or we may take first the extended systems $(\mathfrak{Q}'; S')$ and $(\mathfrak{Q}''; S'')$ and form their composite system, which we denote by $\bar{\mathfrak{Q}}; \bar{S})$. It is desirable to compare the two systems $(\mathfrak{Q}; S)$ and $(\bar{\mathfrak{Q}}; \bar{S})$ thus derived. Let it be assumed that the systems $(\mathfrak{P}'; R')$ and $(\mathfrak{P}''; R'')$ satisfy the seven postulates; then the systems $(\mathfrak{Q}; S)$ and $(\bar{\mathfrak{Q}}; \bar{S})$ also satisfy them. The class $\bar{\mathfrak{Q}}$ consists of all elements of the four forms

$$(1)\ \bar{q} = p' p'', \qquad (2)\ \bar{q} = u' u'', \qquad (3)\ \bar{q} = p' u'', \qquad (4)\ \bar{q} = u' p''.$$

* It is necessary to exclude the trivial case in which the class $\mathfrak{P}$ has no elements.

Now all elements $p = p' p''$ belong to $\mathfrak{P}$ and are therefore in $\mathfrak{Q}$; also, since all classes u' and u'' have properties 1–7, every class $u = u' u''$ has properties 1–7 and is an ideal element of $(\mathfrak{P}; R)$ and consequently appears in $\mathfrak{Q}$. Further, it is clear that a class v of the form $v'_{p'} u''$ or $u' v''_{p''}$ has properties 1–7 and is therefore an element of $\mathfrak{Q}$.

Conversely, every element of $\mathfrak{Q}$ is either a p or a u; every p is of the form $p = p' p''$ and therefore is in $\mathfrak{Q}$; every u is a class having properties 1–7, hence, by lemma II above, is of the form $u = v' v''$; and by lemma I above v' and v'' have properties 1–6; therefore by theorem I v' is a $v'_{p'}$ or a u, and v'' is a $v''_{p''}$ or a u'', so that every u of $\mathfrak{Q}$ is of the form $u' u''$ or $v'_{p'} u''$ or $u' v''_{p''}$.

We arrive, then, at the following theorem:

THEOREM V. *If from two systems, $(\mathfrak{P}'; R')$ and $(\mathfrak{P}''; R'')$, which satisfy the seven postulates we derive $(\mathfrak{Q}; S)$ by composition then extension, and $(\overline{\mathfrak{Q}}; \overline{S})$ by extension then composition, the two systems $(\mathfrak{Q}; S)$ and $(\overline{\mathfrak{Q}}; \overline{S})$ are related as follows:*

(a) *The elements q of $\mathfrak{Q}$ are in reciprocal one-to-one correspondence with the elements $\overline{q}$ of $\overline{\mathfrak{Q}}$ in such manner that, if q corresponds to $\overline{q}$, then $q = p' p''$ and $\overline{q} = p' p''$, or q is of the form $u = u' u''$, where $\overline{q} = u' u''$, or of the form $u = v'_{p'} u''$, where $\overline{q} = p' u''$, or of the form $u = u' v''_{p''}$, where $\overline{q} = u' p''$.*

(b) *If under the correspondence of* (a) *q corresponds to $\overline{q}$, then the classes $\mathfrak{S}$ such that $\mathfrak{S} S q$ are in reciprocal one-to-one correspondence with the classes $\overline{\mathfrak{S}}$ such that $\overline{\mathfrak{S}} \overline{S} \overline{q}$ in such manner that, if $\mathfrak{S}$ corresponds to $\overline{\mathfrak{S}}$, then under the correspondence of* (a) *the elements of $\mathfrak{S}$ correspond to elements of $\overline{\mathfrak{S}}$, and those elements of $\overline{\mathfrak{S}}$ which do not correspond to elements of $\mathfrak{S}$ are of the form $p' u''$ or $u' p''$.*

For a given $\overline{q}$ a class $\overline{\mathfrak{S}}$ such that $\overline{\mathfrak{S}} \overline{S} \overline{q}$ must be of the form $\overline{\mathfrak{S}} = \mathfrak{S}' \mathfrak{S}''$, and if $\mathfrak{S}'$ contains an element p' for which there is no subclass $\mathfrak{R}'$ of $\mathfrak{S}'$ such that $\mathfrak{R}' R' p'$, it is obvious that for every u'' in $\mathfrak{S}''$ the element $p' u''$ is in $\overline{\mathfrak{S}}$, while the corresponding element $v'_{p'} u''$ is not in the class $\mathfrak{S}$ that corresponds to $\overline{\mathfrak{S}}$ under the correspondence of (b).*

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* An important fact to be noted here is that it is with respect to the relations S and $\overline{S}$ that this discrepancy appears. Our purpose in defining a relation S and an extended system $(\mathfrak{Q}; S)$ is to show more completely than would otherwise be possible the operation of our definition of ideal elements. We make no further use of relations S, but treat ideal elements u as associated with systems $(\mathfrak{P}; R)$. A considerable simplification is introduced in Chapter II. It is sufficient for our purpose that if $\mathfrak{U}'$ is the class of ideal elements arising in $(\mathfrak{P}'; R')$ and $\mathfrak{U}''$ is the class of ideal elements arising in $(\mathfrak{P}''; R'')$, then the class $\mathfrak{U}$ of ideal elements arising in the composite system $(\mathfrak{P}; R)$ may be regarded as identical with the sum of the composite classes $\mathfrak{U}' \mathfrak{U}''$, $\mathfrak{U}' \mathfrak{P}''$ and $\mathfrak{P}' \mathfrak{U}''$. And this is seen to be permissible, in the sense that $\mathfrak{U}$ consists of all classes u of the form $u' u''$ or $u' v''_{p''}$ or $v'_{p'} u''$.

Consider a set of r systems, $(\mathfrak{P}^1; R^1)$, $(\mathfrak{P}^2; R^2)$, ..., $(\mathfrak{P}^r; R^r)$. From the first two we may construct the composite system $(\mathfrak{P}^{1,2}; R^{1,2})$, and from this system and $(\mathfrak{P}^3; R^3)$ we obtain the composite system $(\mathfrak{P}^{1,2,3}; R^{1,2,3})$. Continuing in this way we arrive at what may be called the iterated composite system $(\mathfrak{P}^{1,\ldots,r}; R^{1,\ldots,r})$. From the definition of the composite of two given systems it is at once evident that this iterated composite system is as follows: $\mathfrak{P}^{1,\ldots,r}$ is the class of all elements of the form $p = p^1 p^2 \ldots . p^r$, where $p^1, p^2, \ldots,$ and p^r belong to the respective classes $\mathfrak{P}^1, \mathfrak{P}^2, \ldots, \mathfrak{P}^r$. That is, $\mathfrak{P}^{1,\ldots,r} = \mathfrak{P}^1 \mathfrak{P}^2 \ldots . \mathfrak{P}^r$. If $\mathfrak{R}_1^{1,\ldots,r}$ and $\mathfrak{R}_2^{1,\ldots,r}$ are subclasses of $\mathfrak{P}^{1,\ldots,r}$, then the relation $\mathfrak{R}_1^{1,\ldots,r} R^{1,\ldots,r} \mathfrak{R}_2^{1,\ldots,r}$ holds if and only if there exist classes $\mathfrak{R}_1^1, \mathfrak{R}_1^2, \ldots, \mathfrak{R}_1^r$ and $\mathfrak{R}_2^1, \mathfrak{R}_2^2, \ldots, \mathfrak{R}_2^r$ such that $\mathfrak{R}_1^{1,\ldots,r} = \mathfrak{R}_1^1 \mathfrak{R}_1^2 \ldots . \mathfrak{R}_1^r$ and $\mathfrak{R}_2^{1,\ldots,r} = \mathfrak{R}_2^1 \mathfrak{R}_2^2 \ldots . \mathfrak{R}_2^r$ and the relations $\mathfrak{R}_1^1 R^1 \mathfrak{R}_2^1$, $\mathfrak{R}_1^2 R^2 \mathfrak{R}_2^2$, ..., $\mathfrak{R}_1^r R^r \mathfrak{R}_2^r$ are all fulfilled.

It is clear that any other iterated composite system derived from this same set of r systems taken in different order can differ from the one considered in notation only. It will be observed, also, that if we form the composites of groups of systems into which this set of r systems may be divided, and then take the composite of these composites, we arrive at a composite system differing only in notation from the iterated composite system first considered. Thus we see that, aside from possible differences of notation, there is a unique composite system of any finite number of systems of the type $(\mathfrak{P}; R)$.

An obvious generalization of theorem IV is:

THEOREM VI. *The composite system of a finite number of systems satisfies the seven postulates if and only if every component system satisfies them.*

As a partial generalization of theorem V we have:

THEOREM VII. *If* $(\mathfrak{P}; R)$ *is the composite of the systems* $(\mathfrak{P}^1; R^1)$, $(\mathfrak{P}^2; R^2)$, ..., $(\mathfrak{P}^r; R^r)$, *then all elements of the form* $q = q^1 q^2 \ldots . q^r$, *where at least one* q^i *is an ideal element of the corresponding system, and the remainder are elements of the corresponding classes* $\mathfrak{P}^i$, *may be regarded as ideal elements of the system* $(\mathfrak{P}; R)$, *in the sense that the class* $u = v_{q^1}^1 v_{q^2}^2 \ldots . v_{q^r}^r$ (*in which, if* q^i *is a* u^i, $v_{u^i}^i$ *is understood to be identical with the class* u^i) *has the properties* 1–7. *And every ideal element of the composite system is a composite class of the type indicated.*

CHAPTER II.

PROPERTIES OF A SYSTEM FOR WHICH IDEAL ELEMENTS ARE DEFINED.

§ 5. *Postulates for a System* ($\mathfrak{P}$; $\mathfrak{U}$; T), *an Instance of which is Associated with Every System* ($\mathfrak{P}$; R).

We have seen in Chapter I that the postulates I–VII on the system ($\mathfrak{P}$; R) permit of an effective definition of ideal elements for the system, and that the postulates persist under the process of composition, this process being suitably defined. A body of postulates more simple in form and more general in application, yet adequate for a development of a theory of multiple and iterated limits, may be stated for a system for which a definition of ideal elements is assumed to exist independently. Special instances of systems may arise, where ideal elements may be defined in terms of special features of the system in such manner that the enlarged system shall possess all the properties that are essential for the application of our general theory of functions, but where it may be very difficult or impossible to treat the system as a special instance of a system ($\mathfrak{P}$; R) satisfying our postulates.* For this reason, as well as for the sake of simplicity, we specify here the conditions on which we rely for the development of the following theory.

The system consists of a class $\mathfrak{P}$ of elements p, a class $\mathfrak{U}$ of ideal elements u, and a relation T between subclasses $\mathfrak{R}$ of $\mathfrak{P}$ and elements of the class $\mathfrak{Q} = \mathfrak{P} + \mathfrak{U}$. The notation for this system is ($\mathfrak{P}$; $\mathfrak{U}$; T), but the symbol T is largely suppressed in practice. That a given subclass $\mathfrak{R}$ has the relation T to a given element q is indicated by $\mathfrak{R}^q$, while $\mathfrak{R}^{-q}$ indicates that the relation T does not hold between $\mathfrak{R}$ and q. A relation T is said to be defined for the classes $\mathfrak{P}$ and $\mathfrak{U}$ when a criterion exists determining for every $\mathfrak{R}$ and every q whether $\mathfrak{R}^q$ or $\mathfrak{R}^{-q}$.

Following are the postulates: †

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* Instances of this sort are found in the systems ($\mathfrak{P}$; K_2) of § 17 and ($\mathfrak{P}$; V) of § 18.

† We discriminate notationally between the present postulates and those of § 2 by the use of parentheses with the numerals. The postulates may be read as follows:

(I) If $\mathfrak{R}$ has the relation T to p, then p is an element of $\mathfrak{R}$.

(II) Every $\mathfrak{R}$ having the relation T to an ideal element u contains at least one element p.

(III) For every q there exists a sequence $\{\mathfrak{R}_n\}$ of classes having the relation T to q such that for every $\mathfrak{R}$ having the relation T to q there exists a number $n_{\mathfrak{R}}$ such that for $n > n_{\mathfrak{R}}$ $\mathfrak{R}_n$ is a subclass of $\mathfrak{R}$.

(VI) For every $\mathfrak{R}$ having the relation T to an element q there exists another class $\mathfrak{R}_1$ having the relation T to q such that for every p in $\mathfrak{R}_1$ there is a subclass $\mathfrak{R}_2$ of $\mathfrak{R}$ having the relation T to p.

(V) If q_1 is distinct from q_2 there exist classes $\mathfrak{R}_1$ and $\mathfrak{R}_2$ having the relation T to q_1 and q_2 respectively such that there is no element common to $\mathfrak{R}_1$ and $\mathfrak{R}_2$.

(I) $\mathfrak{R}^p \,.\supset.\, p^{\mathfrak{R}}$.

(II) $\mathfrak{R}^u \,.\supset.\, \exists\, p^{\mathfrak{R}}$.

(III) $q \,.:\supset:.\, \exists\, \{\mathfrak{R}_n\} \ni [(n \,.\supset.\, \mathfrak{R}_n^q) \,.\, (\mathfrak{R}^q :\supset: \exists\, n_{\mathfrak{R}} \ni n > n_{\mathfrak{R}} \,.\supset.\, \mathfrak{R}_n^{\mathfrak{R}})]$.

(IV) $R^q :\supset: \exists\, \mathfrak{R}_1^q \ni (p^{\mathfrak{R}_1} \,.\supset.\, \exists\, \mathfrak{R}_2^{\mathfrak{R}} \ni \mathfrak{R}_2^p)$.

(V) $q_1 \neq q_2 \,.\supset.\, \exists\, (\mathfrak{R}_1^{q_1} \,.\, \mathfrak{R}_2^{q_2}) \ni \overline{\exists}\, p \ni p^{\mathfrak{R}_1} \,.\, p^{\mathfrak{R}_2}$.

A system $(\mathfrak{P}; \mathfrak{U}; T)$ may be derived from a system $(\mathfrak{P}; R)$ as follows: Let $\mathfrak{P}$ and $\mathfrak{U}$ of the system $(\mathfrak{P}; \mathfrak{U}; T)$ be respectively the class $\mathfrak{P}$ of the system $(\mathfrak{P}; R)$ and the class of ideal elements arising in $(\mathfrak{P}; R)$; and let a subclass $\mathfrak{R}$ have the relation T to an element p if and only if $\mathfrak{R} R p$, and let $\mathfrak{R}$ have the relation T to an ideal element u if and only if $\mathfrak{R}$ is a member of the class u which constitutes the ideal element of the system $(\mathfrak{P}; R)$.

It may easily be seen that if the system $(\mathfrak{P}; R)$ satisfies the seven postulates of § 2 the resulting system $(\mathfrak{P}; \mathfrak{U}; T)$ satisfies the five postulates stated above. By the mediation of the foregoing definition of T in terms of R, either example 0 of § 2, or the example of § 3, may serve to establish the consistency of the present postulates; and in this same way examples 1, 3, 4 and 7 of § 2 serve as instances of systems satisfying respectively all but (I), all but (III), all but (IV) and all but (V). To complete the proof of the independence of the five postulates we have only to show a system failing to satisfy (II) but satisfying the remaining four postulates. Such a system is the following: $\mathfrak{P}$ is the class of positive integers, and $\mathfrak{U}$ consists of a single ideal element u. The relation $\mathfrak{R}^p$ holds if and only if $\mathfrak{R}$ contains only the single element p, while $\mathfrak{R}^u$ holds if and only if $\mathfrak{R}$ is the null class.

From two systems, $(\mathfrak{P}'; \mathfrak{U}'; T')$ and $(\mathfrak{P}''; \mathfrak{U}''; T'')$, we derive a composite system $(\mathfrak{P}; \mathfrak{U}; T)$, where $\mathfrak{P} = \mathfrak{P}'\mathfrak{P}''$ and $\mathfrak{U}$ consists of all elements of the form $u'u''$ or $u'p''$ or $p'u''$, *i. e.*, $\mathfrak{U} = \mathfrak{U}'\mathfrak{U}'' + \mathfrak{U}'\mathfrak{P}'' + \mathfrak{P}'\mathfrak{U}''$, so that if $\mathfrak{Q}' = \mathfrak{P}' + \mathfrak{U}'$ and $\mathfrak{Q}'' = \mathfrak{P}'' + \mathfrak{U}''$ and $\mathfrak{Q} = \mathfrak{P} + \mathfrak{U}$, then $\mathfrak{Q}$ is the product or composite class $\mathfrak{Q}'\mathfrak{Q}''$. The relation $\mathfrak{R}^q$ holds if and only if there exist classes $\mathfrak{R}'$ and $\mathfrak{R}''$, subclasses of $\mathfrak{P}'$ and $\mathfrak{P}''$ respectively, such that $\mathfrak{R} = \mathfrak{R}'\mathfrak{R}''$, and $\mathfrak{R}'^{q'}$ and $\mathfrak{R}''^{q''}$, where $q = q'q''$. Obviously, this definition of the composite system is consistent with the definition employed in § 4 and the above definition of T in terms of R.

Analogous to theorem IV of Chapter I is the following theorem, the proof of which should cause no difficulty:

THEOREM I. *The composite system* $(\mathfrak{P}; \mathfrak{U}; T)$ *of two systems* $(\mathfrak{P}'; \mathfrak{U}'; T')$ *and* $(\mathfrak{P}''; \mathfrak{U}''; T'')$ *satisfies the postulates* (I)–(V) *if and only if both component systems satisfy them.*

As in § 4, the definition of the composite of two systems leads to a unique composite system for any finite number of systems of the type considered. We clearly have here a theorem analogous to theorem VI of Chapter I.

§ 6. *Limiting Elements: Postulates of F. Riesz.*

The notion of limiting element of a subclass is of primary importance for the purpose in hand. From the point of view of the present investigation, it is sufficient to define limiting elements only for subclasses* $\mathfrak{R}$ of $\mathfrak{P}$.

Def. 1. An element q is a *limiting element* of the subclass $\mathfrak{R}_1$ of $\mathfrak{P}$ if every $\mathfrak{R}$ such that $\mathfrak{R}^q$ contains an element of $\mathfrak{R}_1$ distinct from q, *i. e.*, if

$$\mathfrak{R}^q \,.\supset .\, \exists\, p \neq q \ni p^{\mathfrak{R}_1} \,.\, p^{\mathfrak{R}}.$$

If q is a p, it is an *actual* limiting element, and if a u, it is an *ideal* limiting element; and if q is an element of $\mathfrak{R}_1$, it is a *proper* limiting element of $\mathfrak{R}_1$, and if not an element of $\mathfrak{R}_1$, it is an *improper* limiting element of $\mathfrak{R}_1$. Evidently, a proper limiting element is always actual, and an ideal limiting element is always improper, but a limiting element may be for the same $\mathfrak{R}_1$ both actual and improper.

In his paper† on "Stetigkeitsbegriff und abstracte Mengenlehre" before the International Congress of Mathematicians at Rome, 1908, F. Riesz proposed a set of postulates on which to build, for an abstract class, a generalization of the theory of point-sets. He first assumes an abstract system which we may denote by $(\mathfrak{P}; C)$, where $\mathfrak{P}$ is a class of elements p, and C is a relation between subclasses $\mathfrak{R}$ of $\mathfrak{P}$ and individual elements p, in the sense that p is a limiting element or element of condensation (Verdichtungstelle) of the class $\mathfrak{R}$. It is of interest here to note that if the abstract class of Riesz be identified with our class $\mathfrak{P}$, and if the relation C be assumed to hold if and only if the element p is a limiting element of $\mathfrak{R}$ by definition 1, then the resulting system is found to satisfy the postulates of Riesz. We establish this fact by proving the following theorem, the four propositions of the theorem being somewhat more general than the four conditions necessary to secure the postulates.

THEOREM II. (a) *Every limiting element of $\mathfrak{R}$ is a limiting element of every class $\mathfrak{R}_1$ containing $\mathfrak{R}$.*

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* To define limiting elements for a general subclass of $\mathfrak{Q}$ it would be necessary to resort to a situation analogous to that of § 3; but since the extended system $(\mathfrak{Q}; S)$ is of the same character as $(\mathfrak{P}; R)$, we should thus revert essentially to the situation found in the special case when the class $\mathfrak{U}$ of the system $(\mathfrak{P}; \mathfrak{U}; T)$ contains no elements. This special case furnishes a close analogue to the usual method of procedure in the matter of limits.

† *Atti*, etc., pp. 18–24.

(b) *If* $\mathfrak{R} = \mathfrak{R}_1 + \mathfrak{R}_2$, *then every limiting element of* $\mathfrak{R}$ *is a limiting element either of* $\mathfrak{R}_1$ *or of* $\mathfrak{R}_2$.

(c) *Only infinite classes have limiting elements.*

(d) *Each limiting element of* $\mathfrak{R}$ *is uniquely determined by the totality of all subclasses of* $\mathfrak{R}$ *of which it is a limiting element.*

Proof: Proposition (a) is immediately evident from definition 1.

(b): Let q be a limiting element of $\overline{\mathfrak{R}} = \overline{\mathfrak{R}}_1 + \overline{\mathfrak{R}}_2$. We are to show that q is a limiting element of $\overline{\mathfrak{R}}_1$ or of $\overline{\mathfrak{R}}_2$. By postulate (III) we have

(1) $$\exists \{\mathfrak{R}_n\} \ni [(n \,.\supset.\, \mathfrak{R}_n^q) \,.\, (\mathfrak{R}^q :\supset: \exists\, n_{\mathfrak{R}} \ni n > n_{\mathfrak{R}} \,.\supset.\, \mathfrak{R}_n^{\mathfrak{R}})];$$

then by definition 1 we see that

(2) $$n \,.\supset.\, \exists\, p_n \ni p_n \neq q \,.\, p_n^{\mathfrak{R}_n} \,.\, p_n^{\overline{\mathfrak{R}}}.$$

A sequence $\{p_n\}$ thus secured satisfies the condition

(3) $$\mathfrak{R}^q :\supset: \exists\, n_{\mathfrak{R}} \ni n > n_{\mathfrak{R}} \,.\supset.\, p_n^{\mathfrak{R}}.$$

Since either $\overline{\mathfrak{R}}_1$ or $\overline{\mathfrak{R}}_2$ must contain an infinite subsequence of $\{p_n\}$, we may suppose that $\overline{\mathfrak{R}}_1$ contains the sequence $\{p_{n_m}\}$, where, if $m_1 \neq m_2$, then $n_{m_1} \neq n_{m_2}$. For a given $\mathfrak{R}$ only a finite number of terms of the sequence $\{p_{n_m}\}$ can precede the term $p_{n_\mathfrak{R}}$ in the sequence $\{p_n\}$; hence by (3) we have

$$\mathfrak{R}^q \,.\supset.\, \exists\, m \ni p_{n_m}^{\mathfrak{R}},$$

and since by (2) every p_{n_m} is distinct from q, we see that q is a limiting element of $\overline{\mathfrak{R}}_1$.

(c): Let q be a limiting element of $\overline{\mathfrak{R}}$. If possible, let $\overline{\mathfrak{R}}$ consist of a finite set of elements, $p_1, p_2, \ldots, p_n$. By postulate (V) we see that for every element p_i distinct from q there exists a class $\mathfrak{R}_i$ not containing p_i such that $\mathfrak{R}_i^q$. By an application of postulate (III) we secure a class $\mathfrak{R}$ such that $\mathfrak{R}^q$, which is a common subclass of all the classes $\mathfrak{R}_i$. This class $\mathfrak{R}$ clearly contains no element of $\overline{\mathfrak{R}}$ distinct from q; hence we reach a contradiction.

(d): We are to show that if q_1 and q_2 are distinct limiting elements of $\overline{\mathfrak{R}}$, then there is a subclass of $\overline{\mathfrak{R}}$ that has one of these as limiting element but not the other. By postulate (V)

$$\exists\, (\mathfrak{R}_1^{q_1} \,.\, \mathfrak{R}_2^{q_2}) \ni {}^{-}\exists\, p \ni p^{\mathfrak{R}_1} \,.\, p^{\mathfrak{R}_2}.$$

Denote by $\overline{\mathfrak{R}}_1$ the greatest common subclass of $\overline{\mathfrak{R}}$ and $\mathfrak{R}_1$; then it is clear that q_2 is not a limiting element of $\overline{\mathfrak{R}}_1$. For a given $\mathfrak{R}$ such that $\mathfrak{R}^{q_1}$ there exists by

postulate (III) a common subclass $\mathfrak{R}_3$ of $\mathfrak{R}$ and $\mathfrak{R}_1$ such that $\mathfrak{R}_3^{q_1}$. Since q_1 is a limiting element of $\overline{\mathfrak{R}}$, $\mathfrak{R}_3$ must contain an element p of $\overline{\mathfrak{R}}$, distinct from q_1, and this element p is obviously in $\overline{\mathfrak{R}}_1$. Thus every $\mathfrak{R}$ such that $\mathfrak{R}^{q_1}$ contains an element of $\overline{\mathfrak{R}}_1$ distinct from q_1; that is, q_1 is a limiting element of $\overline{\mathfrak{R}}_1$.

The four postulates of F. Riesz are equivalent to the four propositions of this theorem if we restrict limiting elements q to actual limiting elements p, and if in (b) $\mathfrak{R}_1$ and $\mathfrak{R}_2$ have no common elements.

For the purpose of introducing ideal elements into the abstract class $\mathfrak{P}$, Riesz considers a system which we may denote by $(\mathfrak{P}; V)$, where V is a relation between subclasses $\mathfrak{R}$ of $\mathfrak{P}$ of the same type as our relation R. He postulates for the system $(\mathfrak{P}; V)$ four properties as follows:

(1) If $\mathfrak{R}_1$ and $\mathfrak{R}_2$ have the relation V, and $\mathfrak{R}_3$ contains $\mathfrak{R}_1$ and $\mathfrak{R}_4$ contains $\mathfrak{R}_2$, then $\mathfrak{R}_3$ and $\mathfrak{R}_4$ have the relation V.

(2) If $\mathfrak{R}_1$ and $\mathfrak{R}_2$ have the relation V, and $\mathfrak{R}_1$ is divided into two classes, then at least one of these has the relation V to $\mathfrak{R}_2$.

(3) Two singular subclasses can not have the relation V.

(4) If $\mathfrak{R}_1$ and $\mathfrak{R}_2$ both have the relation V to a given singular class p, then they have the relation V to each other.

A definition of C in terms of V is given by Riesz, by which the relation C holds for a class $\mathfrak{R}$ and an element p if and only if the class $\mathfrak{R}$ and the singular class whose element is p have the relation V. A system $(\mathfrak{P}; C)$ thus obtained from a system $(\mathfrak{P}; V)$ which satisfies the first three conditions above has the first three properties postulated for the system $(\mathfrak{P}; C)$.

From a system $(\mathfrak{P}; \mathfrak{U}; T)$ we obtain a system $(\mathfrak{P}; V)$ as follows: The class $\mathfrak{P}$ of the system $(\mathfrak{P}; V)$ is identical with the class $\mathfrak{P}$ of the system $(\mathfrak{P}; \mathfrak{U}; T)$. The relation V holds between two subclasses of $\mathfrak{P}$ if and only if the two have a common limiting element (actual or ideal) or one subclass contains a limiting element of the other.*

It is easily seen that if the system $(\mathfrak{P}; \mathfrak{U}; T)$ satisfies the five postulates of § 5, then the resulting system $(\mathfrak{P}; V)$ fulfils the four conditions prescribed by Riesz. In fact, the propositions (a), (b) and (c) of theorem II contain sufficient conditions on the system $(\mathfrak{P}; \mathfrak{U}; T)$ to secure this result.

Riesz defines an ideal element as a class v of subclasses $\mathfrak{R}$ which satisfies the following conditions:

(a) If $\mathfrak{R}$ belongs to v and $\mathfrak{R}_1$ contains $\mathfrak{R}$, then $\mathfrak{R}_1$ belongs to v.

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* Compare Riesz, *loc. cit.*, p. 23.

(b) If $\mathfrak{R}$ belongs to v and consists of two subclasses, $\mathfrak{R}_1$ and $\mathfrak{R}_2$, then either $\mathfrak{R}_1$ or $\mathfrak{R}_2$ belongs to v.

(c) Every two classes $\mathfrak{R}_1$ and $\mathfrak{R}_2$ of v have the relation V.

(d) The class v is not contained in any different class v_1 having properties (a), (b) and (c).

(e) No element p is contained in every $\mathfrak{R}$ of v or has the relation V to every $\mathfrak{R}$ of v.

A special case of theorem II, (d), is the proposition: Every ideal element u is uniquely determined by the totality of all subclasses of $\mathfrak{P}$ of which it is a limiting element. It is not difficult to see that such a totality of classes for a given u constitutes a class v fulfilling the five conditions just given. In fact, condition (a) follows from proposition (a) of theorem II, condition (b) from (b) of theorem II, condition (c) from the definition of V in terms of T, and conditions (d) and (e) from (d) of theorem II. Thus every element of the class $\mathfrak{U}$ of the system $(\mathfrak{P}; \mathfrak{U}; T)$ corresponds uniquely to an ideal element of the system $(\mathfrak{P}; V)$.

It may be observed that by the mediation of the definition of a system $(\mathfrak{P}; \mathfrak{U}; T)$ in terms of a system $(\mathfrak{P}; R)$ there is associated with every system $(\mathfrak{P}; R)$ a definite system $(\mathfrak{P}; V)$, and that if the former satisfies the seven postulates of § 2, the latter must fulfil the conditions stated by Riesz. It is clear, also, that every ideal element arising in the system $(\mathfrak{P}; R)$ by our definition corresponds uniquely to an ideal element arising in the associated system $(\mathfrak{P}; V)$ by the definition given by Riesz.

§ 7. *The Fréchet Limit: Properties of Classes.*

In his thesis, "Sur quelques points du calcul fonctionnel," Paris, 1906, M. Fréchet* makes use of an undefined relation between sequences of elements and individual elements. By imposing certain conditions on this relation he is able to develop a theory, analogous to the theory of point-sets and of continuous functions, in which an element that has the undefined relation to a sequence plays the rôle of limit of the sequence. It is of advantage here to show that the notion of limit of a sequence of elements as defined below satisfies the conditions stated by Fréchet.

Def. 2. The sequence $\{p_n\}$ has the limit q if and only if for every $\mathfrak{R}$ such that $\mathfrak{R}^q$ there is a term of the sequence such that all following terms are in the class $\mathfrak{R}$. In symbols:†

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* *Rendiconti del Circolo Matematico di Palermo,* Vol. XXII.

† The notation $\lim_{n=\infty} p_n = q$ is here replaced by the more convenient but equally suggestive notation $L_n p = q$. Note that a sequence may have an ideal element as limit.

$$L p_n = q \;.:\equiv:.\; (\{p_n\} \cdot q) \ni (\mathfrak{R}^q : \supset : \exists\, n_{\mathfrak{R}} \ni n > n_{\mathfrak{R}} \cdot \supset \cdot p_n^{\mathfrak{R}}).$$

The conditions stated by Fréchet for the definition of the limit relation between sequence and element are implied by the following theorem:

THEOREM III. (a) *A sequence formed by repeating a single element has that element for limit.*

(b) *A sequence can not have two distinct limits.*

(c) *If a sequence* $\{p_n\}$ *has a limit* q, *then every subsequence* $\{p_{n_m}\}$ *such that* n_m *becomes infinite with* m *has the limit* q.

Proof: Proposition (a) is an immediate consequence of postulate (I).

(b): Suppose a sequence $\{p_n\}$ to have two distinct limits, q_1 and q_2. By postulate (V) we have

$$\exists\, (\mathfrak{R}_1^{q_1} \cdot \mathfrak{R}_2^{q_2}) \ni {}^{-}\exists\, p \ni p^{\mathfrak{R}_1} \cdot p^{\mathfrak{R}_2}.$$

But by the supposition

$$\exists\, n_{\mathfrak{R}_1} \ni n > n_{\mathfrak{R}_1} \cdot \supset \cdot p_n^{\mathfrak{R}_1} \quad \text{and} \quad \exists\, n_{\mathfrak{R}_2} \ni n > n_{\mathfrak{R}_2} \cdot \supset \cdot p_n^{\mathfrak{R}_2}.$$

By considering p_n such that n is greater than both $n_{\mathfrak{R}_1}$ and $n_{\mathfrak{R}_2}$, we reach a contradiction.

(c): By hypothesis we have

$$\mathfrak{R}^q : \supset : \exists\, n_{\mathfrak{R}} \ni n > n_{\mathfrak{R}} \cdot \supset \cdot p_n^{\mathfrak{R}},$$

and since n_m becomes infinite with m there is for every $n_{\mathfrak{R}}$ a number $m_{\mathfrak{R}}$ such that if m is greater than $m_{\mathfrak{R}}$, then n_m is greater than $n_{\mathfrak{R}}$; thus

$$\mathfrak{R}^q : \supset : \exists\, m_{\mathfrak{R}} \ni m > m_{\mathfrak{R}} \cdot \supset \cdot p_{n_m}^{\mathfrak{R}},$$

which is the required condition.

The three propositions of the theorem are equivalent to the properties of the Fréchet limit, except that he uses instead of (c) a less restrictive condition, obtained by adding to the hypothesis of (c) the restriction that the elements of the subsequence are taken in the same order as in the original sequence. In the proof just given we made use only of postulates (I) and (V), so that these two conditions on a system $(\mathfrak{P}; \mathfrak{U}; T)$ are sufficient for the development of a theory at least as extensive as that pertaining to the class (L) of Fréchet.*

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* Fréchet denotes by (L) what we should represent by $(\mathfrak{P}; L)$, where L is a relation between sequences of elements of $\mathfrak{P}$ and individual elements of $\mathfrak{P}$. Observe that the limit relation which we have defined differs in type from that of Fréchet to the extent that we include ideal limiting elements. One sees, however, that the presence of ideal elements does not interfere with the application to the present situation of the theorems proved by Fréchet on the basis of his limit relation and without the aid of his *écart* or his *voisinage*.

THEOREM IV. *A necessary and sufficient condition that q shall be a limiting element of $\mathfrak{R}$ is that there exist a sequence $\{p_n\}$ of distinct elements of $\mathfrak{R}$ such that $L\, p_n = q$.*

It is necessary: By postulate (III) we have

$$\exists\ \{\mathfrak{R}_n\} \ni [(n \,.\supset .\, \mathfrak{R}_n^q) \,.\, (\overline{\mathfrak{R}}^q :\supset: \exists\ n_{\overline{\mathfrak{R}}} \ni n > n_{\overline{\mathfrak{R}}} \,.\supset .\, \mathfrak{R}_n^{\overline{\mathfrak{R}}})],$$

and, q being a limiting element of $\mathfrak{R}$, we have by definition 1,

$$n .\supset .\ \exists\ p_n \neq q \ni p_n^{\mathfrak{R}_n} \,.\, p_n^{\mathfrak{R}}.$$

The sequence $\{p_n\}$ thus secured is such that $L\, p_n = q$, and since the elements of the sequence constitute a class having the limiting element q, we see by theorem II, (c), that the number of distinct elements of the sequence is not finite. There exists, then, an infinite subsequence $\{p_{n_m}\}$ of distinct terms such that n_m becomes infinite with m, and which, by theorem III, (c), has the limit q.

It is sufficient: This proposition is a direct result of proposition (a) of theorem II.

Theorem IV shows that our definition of limiting element of a subclass is consistent with the definition employed by Fréchet. It may be noticed that in establishing these relations with the work of Fréchet, and the relations to the work of Riesz discussed in the previous section, no use has been made of postulate (IV); it is clear, therefore, that while we have sacrificed much in the matter of generality, we gain somewhat in the extent of the theory available for our system. We consider here certain properties of subclasses that are found useful in the next chapter.

Def. 3. The *derived class* of a subclass $\mathfrak{R}$ is the class of all limiting elements of $\mathfrak{R}$.

Def. 4. A subclass is *closed* if it contains its derived class.

Def. 5. A subclass $\mathfrak{R}$ is *compact** if every infinite subclass of $\mathfrak{R}$ has at least one limiting element.

The propositions of the following theorem, which are given by Fréchet, are seen to be valid here, his proof of (d) being entirely applicable to the present situation, and the first three propositions being obvious deductions from the definition of compactness.

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* See Fréchet, *loc. cit.*, p. 6. Here, again, attention must be called to the fact that we recognize ideal limiting elements.

Theorem V. (a) *Every subclass of a compact class is compact.*

(b) *If every subclass of $\mathfrak{R}$ is compact, then $\mathfrak{R}$ is compact.*

(c) *A class formed of a finite number of compact classes is compact.*

(d) *If every member of a sequence $\{\mathfrak{R}_n\}$ of subclasses of a compact class $\mathfrak{R}$ is closed, contains the succeeding member, and contains at least one element, then there is an element common to all classes of the sequence.*

A proposition somewhat different in content from this last, but permitting of a very similar proof, is stated by Riesz,* and may be stated here as follows:

Theorem VI. *If every member of a sequence $\{\mathfrak{R}_n\}$ of infinite subclasses of a compact class $\mathfrak{R}$ contains the succeeding member, then the members of the sequence possess at least one common limiting element.*

An important proposition in the theory of point-sets and in the analogous theories † in the domain of general analysis is the following: "The derived class of every subclass is closed." The following theorem, in the proof ‡ of which we find the first use for postulate (IV), reduces to this proposition in case no ideal elements exist, *i. e.*, in case $\mathfrak{U}$ is the null class.

Theorem VII. *If $\overline{\mathfrak{R}}_1$ is the class of all actual limiting elements of $\overline{\mathfrak{R}}$, then every limiting element of $\overline{\mathfrak{R}}_1$ is a limiting element of $\overline{\mathfrak{R}}$.*

Proof: By postulate (IV) we have for a given limiting element q of $\overline{\mathfrak{R}}_1$,

$$(1) \qquad \mathfrak{R}^q : \supset : \exists\, \mathfrak{R}_1^q \ni p^{\mathfrak{R}_1} . \supset . \exists\, \mathfrak{R}_2^{\mathfrak{R}} \ni \mathfrak{R}_2^p .$$

Now such a class $\mathfrak{R}_1$ must contain an element p of $\overline{\mathfrak{R}}_1$ distinct from q. Then there is a subclass $\mathfrak{R}_2$ of $\mathfrak{R}$ such that $\mathfrak{R}_2^p$, and p being a limiting element of $\overline{\mathfrak{R}}$,

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* *Loc. cit.*, p. 20.

† E. R. Hedrick, "On Properties of a Domain for which Any Derived Set is Closed," *Transactions of the American Mathematical Society*, Vol. XII (1911), p. 289.

‡ Fréchet shows (*loc. cit.*, p. 15) that this proposition does not follow from the hypotheses he has made on the class (L). He secures this theorem only after the introduction of the notion of *voisinage*. The following example shows that the theorem is not a consequence of postulates (I), (II), (III) and (V), which, as we have shown, are together as strong as the postulates on the class (L) combined with the postulates of Riesz. We specify a system $(\mathfrak{P}; \mathfrak{U}; T)$ as follows: $\mathfrak{P}$ consists of an element p, a sequence $\{p_n\}$ of elements, and a double sequence $\{p_{mn}\}$ of elements. Two elements having different notations are distinct. $\mathfrak{U}$ is the null class. The relation $\mathfrak{R}^p$ holds if and only if $\mathfrak{R}$ consists of the element p together with all, excepting a finite number, of the elements of the sequence $\{p_n\}$. For a given n, the relation $\mathfrak{R}^{p_n}$ holds if and only if $\mathfrak{R}$ consists of the element p_n together with all, excepting a finite number, of the elements of the simple sequence $\{p_{mn}\}$. For a given m and n, $\mathfrak{R}^{p_{mn}}$ holds if and only if $\mathfrak{R}$ consists of the single element p_{mn}. This system satisfies postulates (I), (II), (III) and (V). For a given n the element p_n is the limit of the sequence $\{p_{mn}\}$; and the sequence $\{p_n\}$ has the limit p. The subclass $\mathfrak{R}$ consisting of the elements of the double sequence $\{p_{mn}\}$ has for its derived class the class $\mathfrak{R}_1$, which consists of the elements of the sequence $\{p_n\}$. The only limiting element of $\mathfrak{R}_1$ is p, which is not in $\mathfrak{R}_1$. Thus the derived class $\mathfrak{R}_1$ is not closed.

there is by theorem IV a sequence of distinct elements of $\overline{\mathfrak{R}}$ in $\mathfrak{R}_2$, and therefore in $\mathfrak{R}$. Clearly then we have

$$\mathfrak{R}^q \cdot \supset \cdot \exists\, p \neq q \ni p^{\overline{\mathfrak{R}}} \cdot p^{\mathfrak{R}};$$

that is, q is a limiting element of $\overline{\mathfrak{R}}$.

The proofs of the following propositions relative to a composite system should cause no difficulty.

Theorem VIII. *If* $(\mathfrak{P}; \mathfrak{U}; T)$ *is the composite system of the systems* $(\mathfrak{P}^1; \mathfrak{U}^1; T^1)$, $(\mathfrak{P}^2; \mathfrak{U}^2; T^2)$, ..., $(\mathfrak{P}^r; \mathfrak{U}^r; T^r)$, *and if* $\mathfrak{R} = \mathfrak{R}^1 \mathfrak{R}^2 \ldots \mathfrak{R}^r$, *and* $q = q^1 q^2 \ldots q^r$, *the following propositions hold:*

(a) *If, for every* n, $p_n = p_n^1 p_n^2 \ldots p_n^r$, *then* $L p_n = q$ *if and only if for* $i = 1, 2, \ldots, r$ *we have* $L p_n^i = q^i$.

(b) *The element* q *is a limiting element of* $\mathfrak{R}$ *if and only if every* q^i $(i = 1, 2, \ldots, r)$ *is contained in* $\mathfrak{R}^i$ *or is a limiting element of* $\mathfrak{R}^i$, *and at least one of the* q^i *is a limiting element of the corresponding* $\mathfrak{R}^i$.

(c) $\mathfrak{R}$ *is closed if and only if every* $\mathfrak{R}^i$ *is closed* $(i = 1, 2, \ldots, r)$.

(d) $\mathfrak{R}$ *is compact if and only if every* $\mathfrak{R}^i$ *is compact* $(i = 1, 2, \ldots, r)$.

(To be concluded in the April number.)